**Proof-reading**

Check that whether the following method is correct or not.

Given $f\left(x\right)=\sqrt{\frac{x}{1+x^{2}}} , x\geq 0$ , find its

**(a)** optimum point(s) ;

**(b)** point of inflection.

**(a)** Let $f\left(x\right)=\sqrt{g\left(x\right)}, g\left(x\right)=\frac{x}{1+x^{2}}, x\geq 0$

 $g^{'}\left(x\right)=\frac{\left(1+x^{2}\right)\left(1\right)-x\left(2x\right)}{\left(1+x^{2}\right)^{2}}=\frac{1-x^{2}}{\left(1+x^{2}\right)^{2}}=0⟹x=1$ ($x=-1$ is rejected)

 When 0 < x < 1, $g^{'}\left(x\right)<0$ and when x > 1, $g^{'}\left(x\right)>0$.

 By the First Derivative Test, $g\left(x\right)$ is a local max. when $x=1$.

 Local Max. of g(x) = $\frac{1}{1+1^{2}}=\frac{1}{2}$.

 Since $\lim\_{x\to +\infty }f(x)=\lim\_{x\to +\infty }\sqrt{\frac{x}{1+x^{2}}}=\lim\_{x\to +\infty }\sqrt{\frac{1}{\frac{1}{x}+x}}=0$ and f(0)=0.

 Hence, the absolute maximum of f(x) =$\sqrt{g\left(x\right)}$= $\sqrt{\frac{1}{2}}≈0.707107$ when $x=1$.

 ∴ f(0) = 0 is the absolute minimum.

**(b)** $g^{'}\left(x\right)=\frac{1-x^{2}}{\left(1+x^{2}\right)^{2}}=\frac{2-\left(1+x^{2}\right)}{\left(1+x^{2}\right)^{2}}=\frac{2}{\left(1+x^{2}\right)^{2}}-\frac{1}{1+x^{2}}$

 $g^{''}\left(x\right)=-\frac{8 x}{\left(1+x^{2}\right)^{3}}+\frac{2x}{\left(1+x^{2}\right)^{2}}=\frac{-8 x+2x\left(1+x^{2}\right)}{\left(1+x^{2}\right)^{3}}=\frac{2x\left(x^{2}-3\right)}{\left(1+x^{2}\right)^{3}}=0$

 $∴x=0, \pm \sqrt{3}$

 Since $x\geq 0$, and obviously when $x=0$, $g\left(x\right)$ is not a point of inflection, $∴x=\sqrt{3}$.

 Also $g^{''}\left(x\right)$ changes sign when x goes through $\sqrt{3}$.

 Therefore when $g\left(\sqrt{3}\right)$ is a point of inflection.

 f($\sqrt{3}$) =$\sqrt{g\left(\sqrt{3}\right)}$= $\sqrt{\frac{\sqrt{3}}{1+\left(\sqrt{3}\right)^{2}}}=\frac{\sqrt[4]{3}}{2}$

 ∴ Point of inflection is $\left(\sqrt{3},\frac{\sqrt[4]{3}}{2}\right)$.

**(a)** Part (a) is still correct.

 $f\left(x\right)=\sqrt{\frac{x}{1+x^{2}}}$ , $f^{'}\left(x\right)=\frac{1}{2}\sqrt{\frac{1+x^{2}}{x}}\frac{d}{dx}\left(\frac{x}{1+x^{2}}\right)=\frac{1}{2}\sqrt{\frac{1+x^{2}}{x}}\frac{\left(1+x^{2}\right)\left(1\right)-x\left(2x\right)}{\left(1+x^{2}\right)^{2}}=\frac{1}{2}\sqrt{\frac{1+x^{2}}{x}}\frac{1-x^{2}}{\left(1+x^{2}\right)^{2}}=0$

 $⟹x=1$ ($x=-1$ is rejected)

 When 0 < x < 1, $f^{'}\left(x\right)<0$ and when x > 1, $f^{'}\left(x\right)>0$.

 By the First Derivative Test, $f\left(x\right)$ is a local max. when $x=1$.

 Max. of f(x) = $\sqrt{\frac{1}{1+1^{2}}}=\sqrt{\frac{1}{2}}≈0.707107$.

**(b)** Part (b) is **NOT** correct. The differentiation is a bit longer, have patenice.

$f^{'}\left(x\right)=\frac{1}{2}\sqrt{\frac{1+x^{2}}{x}}\frac{1-x^{2}}{\left(1+x^{2}\right)^{2}}=\frac{1}{2}\left(\frac{1-x^{2}}{x^{\frac{1}{2}}\left(1+x^{2}\right)^{\frac{3}{2}}}\right)$

 Put u = $\frac{1}{2}\left(\frac{1-x^{2}}{x^{\frac{1}{2}}\left(1+x^{2}\right)^{\frac{3}{2}}}\right)$ , $u^{2}=\frac{1}{4}\left(\frac{\left(1-x^{2}\right)^{2}}{x\left(1+x^{2}\right)^{3}}\right)$

 $2u\frac{du}{dx}=\frac{x\left(1+x^{2}\right)^{3}\frac{d}{dx}\left(1-x^{2}\right)^{2}-\left(1-x^{2}\right)^{2}\frac{d}{dx}x\left(1+x^{2}\right)^{3}}{4x^{2}\left(1+x^{2}\right)^{6}}=\frac{x\left(1+x^{2}\right)^{3}2\left(1-x^{2}\right)\left(-2x\right)-\left(1-x^{2}\right)^{2}\left[\left(1+x^{2}\right)^{3}+3x\left(1+x^{2}\right)^{2}\left(2x\right)\right]}{4x^{2}\left(1+x^{2}\right)^{6}}$

 2$f^{'}\left(x\right)f^{'}'\left(x\right)=\frac{x\left(1+x^{2}\right)2\left(1-x^{2}\right)\left(-2x\right)-\left(1-x^{2}\right)^{2}\left[\left(1+x^{2}\right)+3x\left(2x\right)\right]}{4x^{2}\left(1+x^{2}\right)^{4}}=\frac{\left(1-x^{2}\right)\left\{-4x^{2}\left(1+x^{2}\right)-\left(1-x^{2}\right)\left[\left(1+x^{2}\right)+3x\left(2x\right)\right]\right\}}{4x^{2}\left(1+x^{2}\right)^{4}}$

 $=\frac{\left(1-x^{2}\right)\left(3 x^{4}-10 x^{2}-1\right)}{4x^{2}\left(1+x^{2}\right)^{4}}$

 $f^{'}'\left(x\right)$ $=\frac{\left(1-x^{2}\right)\left(3 x^{4}-10 x^{2}-1\right)}{4x^{2}\left(1+x^{2}\right)^{4}}×\frac{x^{\frac{1}{2}}\left(1+x^{2}\right)^{\frac{3}{2}}}{1-x^{2}}=\frac{3 x^{4}-10 x^{2}-1}{4x^{\frac{3}{2}}\left(1+x^{2}\right)^{\frac{5}{2}}}$

 $f''\left(x\right)=0⟹3 x^{4}-10 x^{2}-1=0$ $⟹ x^{2}=\frac{5+2\sqrt{7}}{2}$ (x>0) $⟹x= \sqrt{\frac{5+2\sqrt{7}}{2}}$

 When x is slightly smaller than $\sqrt{\frac{5+2\sqrt{7}}{2}}$, $x^{2}$ is slightly smaller than $\frac{5+2\sqrt{7}}{2}$,

 $3 x^{4}-10 x^{2}-1$ is slightly smaller than 0.

 When x is slightly bigger than $\sqrt{\frac{5+2\sqrt{7}}{2}}$, $x^{2}$ is slightly bigger than $\frac{5+2\sqrt{7}}{2}$,

 $3 x^{4}-10 x^{2}-1$ is slightly bigger than 0.

 $f''\left(x\right)$ change sign as x goes through x = $\sqrt{\frac{5+2\sqrt{7}}{2}}$. It is a point of inflection.

 $f\left(\sqrt{\frac{5+2\sqrt{7}}{2}}\right)=\frac{\sqrt[4]{15+6\sqrt{7}}}{\sqrt{2\left(4+\sqrt{7}\right)}}$ , Point of inflection is $\left(\sqrt{\frac{5+2\sqrt{7}}{2}}, \frac{\sqrt[4]{15+6\sqrt{7}}}{\sqrt{2\left(4+\sqrt{7}\right)}}\right)$

 **Taking square root cannot change the optimum points of a curve (if the curve is well defined) but may change the points of inflection of the curve.**

 **Small exercise**

 Find the point of inflection of the curve:

 (a) $y=\left(x-1\right)x\left(x+1\right)+2$ , where $-1<x<1$.

 (b) $y=\sqrt{\left(x-1\right)x\left(x+1\right)+2}$ , where $-1<x<1$.

 (a) $\left(0 , 2\right)$

 (b) $\left(0.04211 , 1.39927\right)$

**8/5/2017**

**Yue Kwok Choy**